Unfinished Learning Series
Math Community of Practice

Session 2:
Assessing and Diagnosing Unfinished Learning in Math

4.NF.A.2
Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model. (Denominators are limited to 2, 3, 4, 5, 6, 8, 10, 12, and 100.)

Standard Analysis Case Study

This is Ms. Hutchins first year teaching fourth grade math at Brightwood Academy. Prior to teaching fourth grade, she taught seventh grade social studies for one year at another school. Ms. Franklin, the grade 4 content lead, is facilitating a planning meeting with the grade 4 team. The grade 4 team is preparing to teach a topic on fraction comparison. Before the meeting Ms. Franklin has requested the teachers review and annotate the grade level standard, 4.NF.A.2 addressed in their upcoming topic.

Ms. Hutchins comes to the meeting prepared with her standard annotations:

Extend understanding of fraction equivalence and ordering.

Conceptual Understanding- students need to reason about fraction size and use understanding of equivalence to compare.

4.NF.A.2

Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model. (Denominators are limited to 2, 3, 4, 5, 6, 8, 10, 12, and 100.)

Strategies

Common numerators?

What are other benchmarks?

? What does this mean?

Number lines, area models

What if students use a common denominator of 24 to compare $\frac{2}{3}$ and $\frac{5}{8}$?

Ms. Franklin: What did you notice this standard was targeting?

Ms. Hutchins: It's targeting fraction comparison.

Mr. Leonard: It's comparing fractions with different numerators and denominators like $\frac{3}{4}$ and $\frac{2}{3}$ by getting common denominators.

Ms. Hutchins: I also noticed the standard named creating common numerators and I wasn't sure what that meant. The way I learned to compare fractions was to find the least common multiple of the denominators to get common denominators. Like 12 is the least common multiple of 4 and 3 so to compare $\frac{3}{4}$ and $\frac{2}{3}$ you just multiply $\frac{3}{4} \times \frac{3}{3} = \frac{9}{12}$ and $\frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$.

Ms. Franklin: That's a really important observation. How do you think students could compare $\frac{3}{4}$ and $\frac{2}{3}$ by getting a common numerator?

Mr. Leonard: Find the least common multiple of the numerators and rename the fractions as equivalent fractions with the same numerator. So, $\frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$ and $\frac{2}{3} \times \frac{3}{3} = \frac{6}{9}$.

Ms. Hutchins: Oh...I see and $\frac{6}{8}$ is greater than $\frac{6}{9}$. Twelfths are easier for me to work with and I noticed the standard limited denominators to 2, 3, 4, 5, 6, 8, 10, 12, and 100. Does that mean we shouldn't have students compare fractions like $\frac{6}{8}$ and $\frac{6}{9}$ because ninths are not included in the standard?

Ms. Franklin: The limit on the denominators is intended to give students time to work with making visual fraction models. While denominators at this grade level are limited to 2, 3, 4, 5, 6, 8, 10, 12, and 100, students may explore other denominators based on strategies used to find common denominators, or numerators but assessment tasks should adhere to the limits set in the standard.

Ms. Hutchins: Okay, so I understand how common numerators and denominators can be used to compare fractions like $\frac{3}{4}$ and $\frac{2}{3}$, but how would students use a benchmark like one-half to compare them?

Ms. Franklin: Let's think about that as a group. What would students need to be able to understand and do to compare $\frac{3}{4}$ and $\frac{2}{3}$ using one-half as a benchmark?

Mrs. Williams: Well they would need to know that $\frac{1}{2} = \frac{2}{4}$ and $\frac{3}{4}$ is greater than $\frac{2}{4}$ so $\frac{3}{4}$ is greater than $\frac{1}{2}$.

Mr. Leonard: Yeah, but $\frac{2}{3}$ is also greater than $\frac{1}{2}$ so they would still need to figure out which fraction was farther from $\frac{1}{2}$. They would need to know how to make number line to see the distance each fraction is from $\frac{1}{2}$.

Ms. Hutchins: Or they could reason $\frac{3}{4}$ is $\frac{1}{4}$ from $\frac{1}{2}$ and $\frac{2}{3} = \frac{4}{6}$ which is only $\frac{1}{6}$ from $\frac{3}{6}$ or $\frac{1}{2}$.

Ms. Franklin: Yes, that would be using both understanding of equivalence and benchmarks to compare fractions. How might students use a benchmark of 1 to compare $\frac{2}{3}$ and $\frac{3}{4}$?

Mrs. Williams: $\frac{3}{4}$ is closer to 1 than $\frac{2}{3}$ so it's the greater fraction.

Ms. Hutchins: You could show them that on a number line. $\frac{2}{3}$ is $\frac{1}{3}$ from 1, and $\frac{3}{4}$ is $\frac{1}{4}$ from 1.

Ms. Franklin: So in addition to $\frac{1}{2}$, 0 and 1 are also helpful benchmarks to compare fractions. How does the standard expect students to demonstrate their knowledge of comparing fractions with different denominators and numerators?

Ms. Hutchins: Students need to use comparison symbols and justify their comparison using visual fraction models. I noticed Topic C uses number lines in almost every lesson.

Mrs. Williams: Yes, the number lines have been really helpful in Topics A and B for students to see equivalent fractions. I think they may struggle to draw their own number lines to place fractions with different denominators on the same number line though.

Mr. Leonard: I think my students would draw two separate number lines. That's what I noticed a lot of them were doing to prove two fractions were equivalent. The problem was when they wouldn't draw the number lines the same length so the equivalent fractions wouldn't line up.

Ms. Franklin: That's a common misconception that's important to address. How does that misconception relate to this standard?

Mrs. Williams: Because they're not using the same size whole, and the standard specifies comparisons are only valid if the whole is the same.

Ms. Hutchins: I wasn't sure what that meant.

Ms. Franklin: Take a look at this standard in the LDOE Companion Document for Teachers to see an example of what this part of the standard is referring to.

Ms. Hutchins: I see it says, "Students must also recognize that they must consider the size of the whole when comparing fractions (i.e., $\frac{1}{2}$ and $\frac{1}{8}$ of two medium pizzas are very different from $\frac{1}{2}$ of one medium and $\frac{1}{8}$ of one large)."

Ms. Franklin: If Mr. Leonard drank $\frac{1}{2}$ of his small bottle water (points to Mr. Leonard's bottle) and I drank $\frac{1}{4}$ of my large bottle of water (holds up a large water bottle), I still drank more water than him because the unit I was drinking from was much larger than his unit.

Ms. Hutchins: I get it now. That seems obvious, but I never learned it that way and I see how it's important for students to recognize the size of a fraction depends on the size of the whole.

Ms. Franklin: Yes, the aspect of rigor this standard is targeting is conceptual understanding. Using visual fraction models is key to helping students understand the significance of the whole when comparing fractions.

Ms. Hutchins: I guess that's why I never learned it. I was just taught to use the LCM to find a common denominator. I never thought about what the whole or fractions I was comparing looked like, it was just a procedure I memorized.

Mr. Leonard: The way I learned it was to cross-multiply using the butterfly method.

Mr. Leonard draws an example of the butterfly method.

Mr. Leonard: I never understood why that worked though.

Ms. Franklin: Those are helpful reflections on what this standard is NOT targeting, just applying an algorithm or using quick tricks to compare fractions doesn't help students make sense of fractions and reason about comparisons. The cluster heading for this standard is "Extend understanding of fraction equivalence and ordering." To know what understanding students are extending, let's look at the prerequisite standards so we can consider the progression of learning this standard is building on, and what it's building to in grade 5. But before we analyze the learning progression for this standard, let's complete an assessment task aligned to 4.NF.A.2 so we have a more clear picture of the standard.

Learning Progression Analysis Protocol

Step 1: Identify the pre-requisite standards connected to the grade level standard in the Nebraska Essential Instructional Guide, and/or using the Achieve the Core Coherence Map.

Step 2: Read the prerequisite standards. Annotate the following…

- Any unfamiliar language or questions you have about the standard
- Aspect of rigor the standard is targeting (conceptual understanding, procedural fluency, application)
- Concept(s) students are expected to understand or know
- What students are expected to do or show
- Strategies and models students are expected to use
- Specifics or limits specified in the standard
- Connections to the grade level standard

Step 3: Read about the prerequisite standards in the LDOE Companion Document for Teachers.

Step 4: Complete assessment tasks aligned to the prerequisite standard.

Step 5: Compare your work with a colleague or the exemplar response. Discuss the following:

- What do you students need to understand and be able to do to be successful on these tasks?
- How do the concepts and skills students need to be successful prepare them for grade level instruction?
- What misconceptions or incomplete understandings may this task reveal?

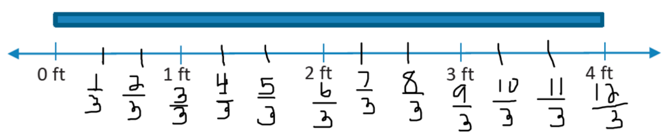
Ms. Hutchins' Prerequisite Standard Annotations & Example Assessment Tasks

4.NF.A.2 Progression Analysis Example

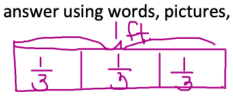
Conceptual. Understanding	Conceptual. Understanding
3.NF.A.3.B Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3. Explain why the fractions are equivalent, e.g., by using a visual fraction model. 4.NF.A.2.- This is foundational for students to generate equivalent fractions with common denominators or numerators to compare and recognize fractions less than or more than half.	3.NF.A.3.C Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers. *Examples: Express 3 in the form 3 = 3/1; recognize that 6/1 = 6; locate 4/4 and 1 at the same point of a number line diagram.* 4.NF.A.2 – This is foundational for students to use whole numbers as benchmarks to compare fractions greater than 1.

Jerry has a fruit roll that is 4 feet long.

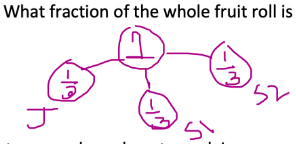
a. Label the number line to show how Jerry might cut his fruit roll into pieces $\frac{1}{3}$ of a foot long. Label every fraction on the number line, including renaming the wholes.



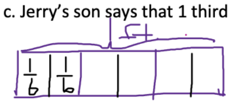
b. Jerry cut his fruit roll into pieces that are $\frac{1}{3}$ of a foot long. Jerry and his 2 sons each eat one piece. What fraction of the whole fruit roll is eaten? Explain your answer using words, pictures, and numbers.



Jerry and his two sons ate $\frac{3}{3}$ or 1 ft. of the fruit roll because 3 copies of $\frac{1}{3}$ are the same as $\frac{3}{3}$.



c. Jerry's son says that 1 third is the same as 2 sixths. Do you agree? Why or why not? Use words, pictures, and numbers to explain your answer



I agree with Jerry's son because dividing each 1 third into 2 smaller equal parts creates sixths. Two-sixths is the same amount of the whole as one-third. One-third and 2 sixths are also the same location on a number line.

Ms. Hutchins' Prerequisite Standard Annotations & Example Assessment Tasks

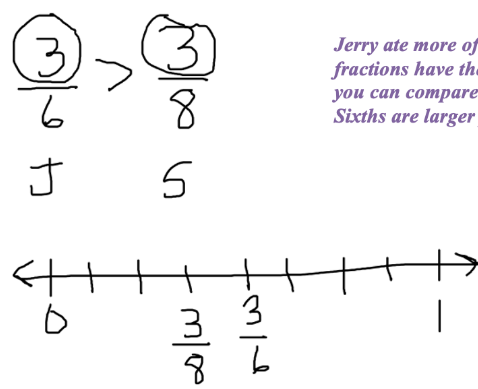
4.NF.A.2 builds on this- Conceptual. Understanding

3.NF.A.3.D

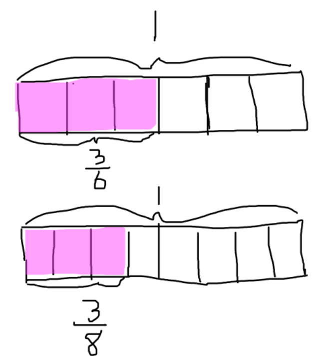
Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

Same language as 4.NF.A.2

Jerry and his son have the exact same granola bars. Jerry has eaten $\frac{3}{6}$ of his granola bar. His son has eaten $\frac{3}{8}$ of his own granola bar. Who has eaten more? Explain your answer using words, pictures, and numbers.



Jerry ate more of the granola bar. Both fractions have the same numerator so you can compare the denominators. Sixths are larger parts than eighths.



Eureka Acceleration Tool
[Grade 4 Module 5 Topic C]

Student One

Diagnostic Assessment: Grade 4
Eureka Module 5, Topic C

Part C: 3.NF.A.3d

7. For the inequality $\frac{1}{2} > \frac{1}{4}$ to be valid, what must be true?

$\frac{1}{2}$ is big part $\frac{1}{4}$ is smaller

8. Complete the sentence with >, =, or <. Explain your thinking and/or justify your choice with a visual fraction model.

$\frac{2}{6}$ < $\frac{5}{6}$

5 is more than 2

9. Complete the sentence with >, =, or <. Explain your thinking and/or justify your choice with a visual fraction model.

$\frac{2}{6}$ < $\frac{2}{8}$

8 is more than 6

Strengths	Unfinished Learning
The student interpreted and used comparison symbols accurately. The student knows one-half is a larger fractional part and one-fourth is a smaller fraction part. This leads me to believe the student has some understanding of unit fractions.	The student work does not include evidence that indicates an understanding that the size of the whole must be equal when comparing fractions. The student applied whole number reasoning to compare the numerators and denominators. This leads me to believe the student may not understand a fraction is a single number and has an emerging understanding of the relationship between the denominator and the size of the fractional parts. The student work does not include any visual fraction models. This leads me to wonder how the student may be visualizing fractions?

Student Two

Diagnostic Assessment: Grade 4
Eureka Module 5, Topic C

Part C: 3.NF.A.3d

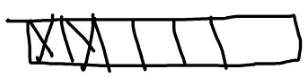
7. For the inequality $\frac{1}{2} > \frac{1}{4}$ to be valid, what must be true?

You have to draw a picture to show your work like a tape model.

8. Complete the sentence with >, =, or <. Explain your thinking and/or justify your choice with a visual fraction model.

$\frac{2}{6}$ < $\frac{5}{6}$

5 out of 6 is more than 2 out of 6.



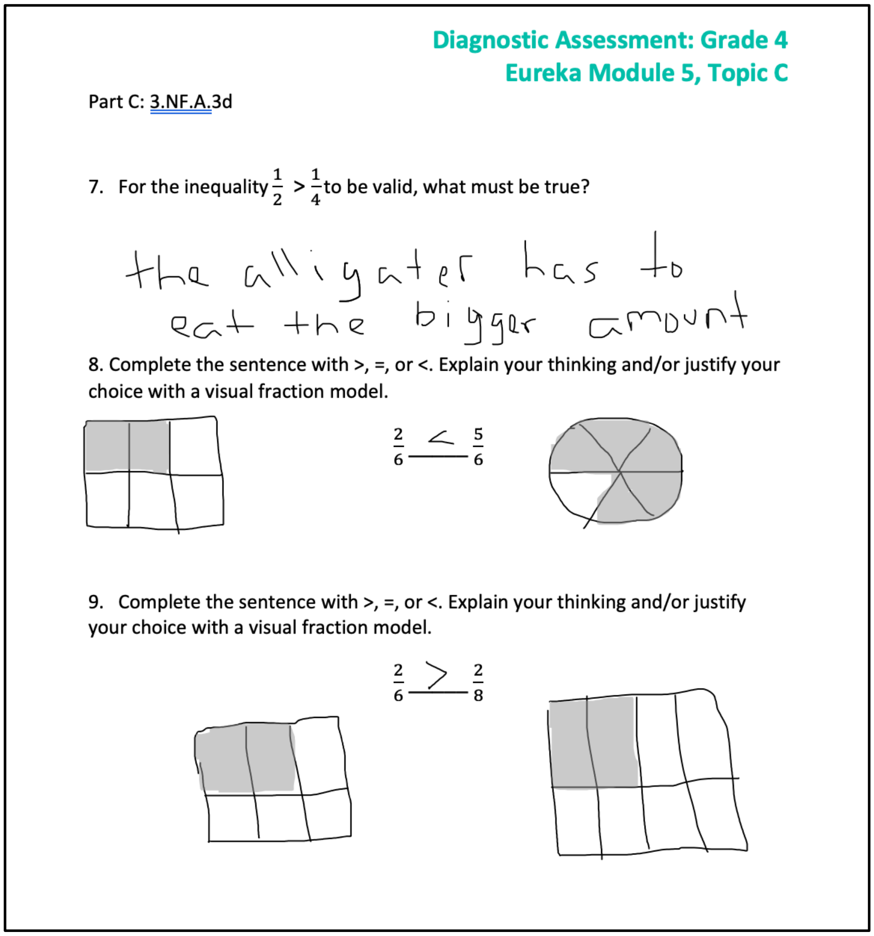
9. Complete the sentence with >, =, or <. Explain your thinking and/or justify your choice with a visual fraction model.

$\frac{2}{6}$ < $\frac{2}{8}$

2 out of 8 is more than 2 out of 6 because 8 has more parts.

Strengths	Unfinished Learning
The student interpreted and used comparison symbols accurately. The student drew tape diagrams to represent fractions. This leads me to believe the student knows… ● fractions can be parts of whole ● fhe denominator represents the number of parts the whole is partitioned into, or the size of the parts ● the numerator represents the number of parts counted, or being considered	The student work shows tape models with different size wholes. This leads me to believe the student does not yet understand that the size of the whole must be equal when comparing fractions. The student tape diagrams show unequal parts. This leads me to believe the student has not yet developed understanding fractions represent equal parts of the whole and/or strategies for equipartitioning. The student used _ out of _ language which leads me to believe the student may not understand a fraction is a single number. The student based his/her comparison for #9 on the number of parts instead of the size of the parts which leads me to believe the student does not yet understand the relationship between the denominator and the size of the fractional parts.

Student Three



Diagnostic Assessment: Grade 4
Eureka Module 5, Topic C

Part C: 3.NF.A.3d

7. For the inequality $\frac{1}{2} > \frac{1}{4}$ to be valid, what must be true?

the alligater has to eat the bigger amount

8. Complete the sentence with >, =, or <. Explain your thinking and/or justify your choice with a visual fraction model.

$\frac{2}{6} < \frac{5}{6}$

9. Complete the sentence with >, =, or <. Explain your thinking and/or justify your choice with a visual fraction model.

$\frac{2}{6} > \frac{2}{8}$

Strengths	Unfinished Learning
The student knows the comparison symbol opens towards the greater value. The student drew different area models to represent fractions. This leads me to believe the student understands fractions can be represented in a variety of ways. The student partitioned the area models into equal parts with some level of precision. This leads me to believe the student understands fractions can represent equal parts of the whole and has developed equipartitioning strategies.	The student work includes evidence that indicates the student has not yet developed understanding that the size of the whole must be equal when comparing fractions. The student drew different size area models. The area models drawn for #9 do not appear to match the comparison the student made. This leads me to wonder if the student understands the magnitude of unit fractions (e.g., one-eighth is a smaller area of the whole than one-sixth).

Student Four

Diagnostic Assessment: Grade 4
Eureka Module 5, Topic C

Part C: 3.NF.A.3d

7. For the inequality $\frac{1}{2} > \frac{1}{4}$ to be valid, what must be true?

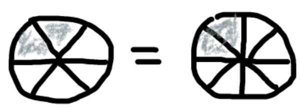
8. Complete the sentence with >, =, or <. Explain your thinking and/or justify your choice with a visual fraction model.

$\frac{2}{6} < \frac{5}{6}$

The denomenater is 6 so i compared 2 and 5.

9. Complete the sentence with >, =, or <. Explain your thinking and/or justify your choice with a visual fraction model.

$\frac{2}{6} = \frac{2}{8}$



Both have 2 peces so they are the same.

Strengths	Unfinished Learning
The student wrote the comparison symbol opening toward the greater amount. This leads me to believe they understand the meaning of the comparison symbols. The student drew different area models and tape diagrams to represent fractions. This leads me to believe the student understands fractions can be represented in a variety of ways. The student partitioned the area models into equal parts with some level of precision. This leads me to believe the student understands fractions can represent equal parts of the whole and has developed equipartitioning strategies.	The student work includes evidence that indicates the student has not yet developed understanding that the size of the whole must be equal when comparing fractions. The student fraction models for #9 and conclusion the fractions are equal leads me to believe the student has an emerging understanding of the relationship between the denominator and the size of the parts. It leads me to believe the student is focusing on the number of parts counted when comparing, rather than the area of the whole a fraction describes.

Ms. Hutchins Data Snapshot

Assessment Task	Got It	Almost Got It	Not Yet
#7	***Evidence of understanding in models and explanation*** Dakari	***Evidence of understanding in models drawn, no explanation*** Janelle, Ivette, Kapone	***No Evidence*** Sydney, Rochelle, Nyla, Byrce, Isaiah, Neveah, Anniyah, Edwin, Joseph, Elijah, Kamal, Malayah, Richard, Jeremiah, Andre, Zion
#8	***Correct comparison and complete reasoning*** Dakari, Janelle, Ivette, Kapone, Rochelle, Nyla	***Correct Comparison, Incomplete Reasoning and/or Inaccurate Model*** Sydney, Isaiah, Neveah, Anniyah, Richard, Zion, Edwin, Elijah	***Incorrect Comparison, and/or Faulty Reasoning*** Byrce, Joseph, Kamal, Malayah, Jeremiah, Andre
#9	***Correct comparison and complete reasoning*** Dakari, Janelle, Ivette, Kapone, Nyla, Elijah, Isaiah	***Correct Comparison, Incomplete Reasoning*** Sydney, Rochelle, Zion	***Incorrect Comparison, and/or Faulty Reasoning*** Byrce, Neveah, Anniyah, Edwin, Joseph, Kamal, Malayah, Richard, Jeremiah, Andre

Strengths	Misconceptions/Unfinished Learning
• Interpretation and use of comparison symbols • Use of tape diagrams and area models to compare fractions • Comparing unit fractions • Understanding the denominator tells the number of equal parts into which a whole is partitioned and the numerator the number of copies of the fractional part • Noticing common numerators	• Not yet recognizing the whole units must be equal for comparisons to be valid • Labeling the whole unit • Applying whole number reasoning to compare fractions (e.g., $\frac{2}{8} > \frac{2}{6}$ because 8 > 6) • Justifying comparisons by reasoning about the denominator and the size of the fractional parts (*as the number of equal parts in a whole (denominator) increases, the size of the fractional parts decreases*)

Let’s Reflect
To what extent is this work currently happening at your school/in your classroom? What implications might this learning have on how you support schools or teachers with assessing and diagnosing unfinished learning in your role?